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# $\overline{\text{OMS}}$ scheme of UV renormalization in the presence of unstable fundamental particles

M.L. Nekrasov

*Institute for High Energy Physics, 142284 Protvino, Russia*

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## Abstract

A generalization of the on-mass-shell scheme of UV renormalization (the  $\overline{\text{OMS}}$  scheme) to the case of presence of unstable fundamental particles is proposed. Its basic ingredients are as follows: (i) the renormalized mass coincides with a real part of the position of the complex pole of the corresponding propagator, (ii) the imaginary part of the on-shell self-energy coincides with the imaginary part of the complex pole position. The latter property implies the gauge-invariance of the imaginary part of the on-shell self-energy in the  $\overline{\text{OMS}}$  scheme and its direct connection with the width of the unstable particle. Starting with the three-loops this connection becomes nontrivial.

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The aim of this Letter is to introduce an effective generalization of the on-mass-shell (OMS) scheme of UV renormalization to the case of presence of unstable fundamental particles. This problem is determined by the difficulties with the gauge invariance, noted in the framework of the conventional generalization of the OMS scheme [1–3] in the cases of W, Z and Higgs bosons beyond the one-loop order [4–7]. In fact, however, even in the case of non-gauge field theories the conventional generalization [1–3] ceases to have those attractive properties which are peculiar to the standard OMS scheme in the case of stable particles. So, finding the “true” generalization of the OMS scheme, possessing the physically-motivated

(and, hence, convenient) properties, is an important task from the general field-theoretic point of view.

Let us begin our analysis with considering the inverse renormalized propagator of a scalar particle, or of  $\delta_{\mu\nu}$ -part of a vector particle. We do not define precisely the sort of particle and the underlying theory since the problem of renormalization is general enough in nature. In terms of the renormalized quantities we have

$$\Delta^{-1}(s) = s - M^2 - \delta Z(s - M^2) - \delta M^2 + \Sigma(s). \quad (1)$$

Here  $M^2$  is the renormalized lagrangian mass,  $\Sigma(s)$  is the self-energy that depends, besides  $s$ , also on  $M^2$  and the renormalized coupling constant  $\alpha$ . Quantities  $\delta M^2$  and  $\delta Z$  describe the counterterm contributions (notice the minus sign in  $\delta Z$  in our notation). Their assigning is to cancel UV divergencies in  $\Sigma(s)$ .

E-mail address: [nekrasov@mx.ihep.su](mailto:nekrasov@mx.ihep.su) (M.L. Nekrasov).

In the framework of perturbation theory this cancellation should be performed order-by-order. So, with

$$\Sigma(s) = \sum_{n=1}^{\infty} \alpha^n \Sigma_n(s), \quad (2)$$

$$\delta Z = \sum_{n=1}^{\infty} \alpha^n C_n^Z, \quad \delta M^2 = \sum_{n=1}^{\infty} \alpha^n C_n^M, \quad (3)$$

the coefficients  $C_n^Z$  and  $C_n^M$  must provide finiteness for  $\Sigma_n(s) - C_n^M - C_n^Z(s - M^2)$ . From the unitarity of the  $S$ -matrix it follows that the counterterms must be real [8]. The operational use of various renormalization schemes confirms that in the commonly used (gauge) theories two real counterterms indeed cancel UV divergences in  $\Sigma_n(s)$ .

It is worth reminding that various renormalization schemes are different in finite parts of counterterms. This difference, in turn, means a different determination of the renormalized lagrangian parameters and the normalization of the Green functions. In the standard OMS scheme the renormalized mass  $M^2$  is made equal to the physical mass  $M_{\text{ph}}^2$  determined by  $\Delta^{-1}(M_{\text{ph}}^2) = 0$ . Besides, the residue at the pole in the propagator is made equal to 1. Both these properties make the OMS scheme very convenient for the practical usage.

In the case of stable particles the above-mentioned properties are provided by the following counterterms:

$$C_n^M = \Sigma_n(M^2), \quad C_n^Z = \Sigma'_n(M^2). \quad (4)$$

However, when the particle under consideration is unstable, this choice of counterterms is not admissible because of the non-vanishing imaginary parts in the self-energy.

The most commonly used way [1–3] of solving the problem consists in replacing (4) by

$$C_n^M = \text{Re } \Sigma_n(M^2), \quad C_n^Z = \text{Re } \Sigma'_n(M^2). \quad (5)$$

However, then the renormalized mass becomes defined by the condition  $\text{Re } \Delta^{-1}(M^2) = 0$ , which does not provide the pole to the propagator. As a result, the renormalized mass becomes no longer physical observable. In the case of electroweak theory this fact manifests itself in the emergence of the gauge-dependence in the renormalized masses of the vector bosons and the Higgs boson [4–7]. This situation is ob-

jectionable and certainly must be cured in a true generalization of the OMS scheme.

Actually, the latter problem has been posed not once. The idea of its solution consists in equating the renormalized mass  $M^2$  to a real part of the position of the complex pole  $s_p$  of the propagator, which is gauge-invariant [4–7,9]. In Ref. [10] this idea has been implemented in a special case of calculation of the two-loop correction to the muon lifetime. However, the general study of the problem has not been made. So, the true generalization of the OMS scheme is still not completed. In particular, the second renormalization condition for  $\Sigma'(s)$  is still not determined properly. The point is that the non-vanishing  $\text{Im } \Sigma'(s)$  prevents the residue in the pole from being equal to 1. In Ref. [10] the second renormalization condition was chosen rather formally, in the form of (5). In the particular case of the two-loop calculation of the muon lifetime this choice did not have adverse consequences. However, on description of the production and decay of unstable particles this choice may lead again to difficulties with gauge invariance (see below).

In the present Letter we propose an unconventional way of fixing the second renormalization condition. It has a clear physical significance, so the name “physical” can be appropriated to this scheme. We call it the  $\overline{\text{OMS}}$  scheme. Under the limit of switching-off the instability, it transforms smoothly to the standard OMS scheme.

The basic point of our consideration is the condition of the gauge-invariance of the position of the complex pole  $s_p$  [4–7,9,11]. Owing to (1) the equation for  $s_p$ , which is  $\Delta^{-1}(s_p) = 0$ , may be rewritten in the form

$$s_p = M^2 + \delta M^2 + \delta Z(s_p - M^2) - \Sigma(s_p). \quad (6)$$

With the aid of (2) and (3) this equation can be solved by iteration. So, denoting the solution up to  $O(\alpha^{n+1})$  correction by  $s_{pn}$ , and introducing the short-card notation  $R_n = \text{Re } \Sigma_n(M^2)$ ,  $I_n = \text{Im } \Sigma_n(M^2)$ , with the primed symbols indicating the derivatives, we get the following sequence of iterative solutions:

$$s_{p0} = M^2, \quad (7)$$

$$s_{p1} = s_{p0} + \alpha(C_1^M - R_1 - iI_1), \quad (8)$$

$$s_{p2} = s_{p1} + \alpha(s_{p1} - s_{p0})(C_1^Z - R'_1 - iI'_1)$$

$$+ \alpha^2 (C_2^M - R_2 - iI_2), \quad (9)$$

$$\begin{aligned} s_{p3} = & s_{p2} + \alpha(s_{p2} - s_{p0})(C_1^Z - R_1' - iI_1') \\ & + \frac{1}{2}\alpha(s_{p1} - s_{p0})^2(-R_1'' - iI_1'') \\ & + \alpha^2(s_{p1} - s_{p0})(C_2^Z - R_2' - iI_2') \\ & + \alpha^3(C_3^M - R_3 - iI_3), \end{aligned} \quad (10)$$

⋮

For methodological reasons we consider, at first, the conventionally generalized OMS scheme [1–3] determined by (5). Then, the listed above solutions are reduced to

$$s_{p0} = M^2, \quad (11)$$

$$s_{p1} = M^2 - i\alpha I_1, \quad (12)$$

$$s_{p2} = M^2 - \alpha^2 I_1 I_1' - i\alpha I_1 - i\alpha^2 I_2, \quad (13)$$

$$\begin{aligned} s_{p3} = & M^2 - \alpha^2 I_1 I_1' - \alpha^3 (I_1 I_2' + I_1' I_2 - \frac{1}{2} I_1^2 R_1'') \\ & - i\alpha I_1 - i\alpha^2 I_2 - i\alpha^3 [I_3 - I_1 (I_1')^2 - \frac{1}{2} I_1^2 I_1''], \end{aligned} \quad (14)$$

⋮

From formulas (11) and (12) we see that in the case of gauge theories the renormalized mass  $M^2$  is gauge-invariant up to  $O(\alpha^2)$  correction. However, the  $O(\alpha^2)$  correction is gauge-dependent since the difference  $M^2 - \text{Re } s_{p2} = \alpha^2 I_1 I_1'$  is like that. This property follows from the gauge-invariance of  $I_1$ , which is the consequence of (12), and the gauge-dependence of  $I_1'$ . The latter property was observed in the case of Z-boson [4,5], W-boson [6], and Higgs boson [7]. So, the gauge-invariance of  $s_p$  implies the gauge-dependence of the renormalized mass  $M^2$  at the two-loop order.<sup>1</sup>

It should be noted that from the viewpoint of underlying principles there is nothing catastrophic in the latter situation, since the renormalized mass is not an observable quantity. However, it is not reasonable to

use in practice such renormalization scheme. A better choice is a scheme where the renormalized mass is gauge-invariant, and it would be even better if the renormalized mass coincided with the observable  $\text{Re } s_p$ .

Now we proceed directly to the construction of the  $\overline{\text{OMS}}$  scheme, paying special attention to the choice of the second renormalization condition. We do that in an iterative manner, order-by-order. So, in the leading order we have  $s_{p0} = M^2$  without alternatives. In the one-loop order we set

$$C_1^M = R_1. \quad (15)$$

Then,  $s_{p1}$  coincides with that of formula (12).

The difference with the conventionally generalized OMS scheme [1–3] appears starting with the two-loop order. Owing to (8), (9) and (15), we have

$$\begin{aligned} s_{p2} = & M^2 - i\alpha I_1 - i\alpha^2 I_1 (C_1^Z - R_1' - iI_1') \\ & + \alpha^2 (C_2^M - R_2 - iI_2). \end{aligned} \quad (16)$$

By assuming,

$$C_1^Z = R_1', \quad (17)$$

we come to the same imaginary part in  $s_{p2}$  as in (13). However, in order to satisfy requirement  $\text{Re } s_p = M^2$ , we have to impose a different condition for  $C_2^M$  (cf. [10]):

$$C_2^M = R_2 + I_1 I_1'. \quad (18)$$

So, taking into account (17) and (18), we obtain

$$s_{p2} = M^2 - i\alpha I_1 - i\alpha^2 I_2. \quad (19)$$

The difference becomes more considerable in the three-loop order. Owing to (10), (12), (17) and (19), we have

$$\begin{aligned} s_{p3} = & s_{p2} - i\alpha^3 I_3 - i\alpha^3 I_1 (C_2^Z - R_2' - \frac{1}{2} I_1 I_1'') \\ & + \alpha^3 (C_3^M - R_3 - I_2 I_1' - I_1 I_2' + \frac{1}{2} I_1^2 R_1''). \end{aligned} \quad (20)$$

Let us note, that the imaginary part of  $s_{p3}$  has a far complicated structure. However, by assuming

$$C_2^Z = R_2' + \frac{1}{2} I_1 I_1'', \quad (21)$$

we can get the simplest possible expression for  $\text{Im } s_{p3}$ , namely  $-i\alpha^3 I_3$ . In order to provide  $\text{Re } s_p = M^2$ , we set

$$C_3^M = R_3 + I_2 I_1' + I_1 I_2' - \frac{1}{2} I_1^2 R_1''. \quad (22)$$

<sup>1</sup> It is interesting to compare this result with that obtained in the framework of pinch-technique approach, which allows UV renormalization after specific rearrangement of contributions to the amplitude. Effectively, this rearrangement means modification of the renormalization scheme. In this approach the gauge-invariance of  $\text{Im } \Sigma(s)$  is maintained, at least at one-loop order (see [12]), so that  $M^2$  is gauge-invariant at the two-loop order. At the higher orders this approach is to be corroborated in the electroweak theory.

As a result, we come to

$$s_{p3} = M^2 - i\alpha I_1 - i\alpha^2 I_2 - i\alpha^3 I_3. \quad (23)$$

The above consideration may be continued up to any  $n$ , providing in the limit  $n \rightarrow \infty$  the following solution:<sup>2</sup>

$$s_p = M^2 - i \operatorname{Im} \Sigma(M^2). \quad (24)$$

Let us summarize the main features of the above construction. At any step  $n$ , when considering the imaginary part of  $s_{pn}$ , we fix the renormalization condition for  $C_{n-1}^Z$  by imposing the requirement  $\operatorname{Im}(s_{pn} - s_{p(n-1)}) = -\alpha^n I_n$ . When considering the real part of  $s_{pn}$ , we fix the renormalization condition for  $C_n^M$  by imposing  $\operatorname{Re} s_{pn} = M^2$ . The divergent contributions to  $C_n^M$  and  $C_{n-1}^Z$  turn out to be  $R_n$  and  $R'_{n-1}$ , respectively. The resulting formulas for  $C_n^M$  and  $C_{n-1}^Z$  can be obtained for any  $n$ . However, the cases with  $n \geq 4$ , most likely, will not be claimed in a foreseeable future. So, we will not be wasting time to find the general solution.

Let us turn now to the discussion.

1. The first question is about the structure of the propagator in the resonance region. By excluding  $\delta M^2$  from (6) in favor of  $s_p$ , one can derive from (1),

$$\begin{aligned} \Delta^{-1}(s) &= (s - s_p)(1 - \delta Z) + \Sigma(s) - \Sigma(s_p) \\ &= (s - s_p)[1 - \delta Z + \Sigma'(s_p)] + O((s - s_p)^2). \end{aligned} \quad (25)$$

From (25) we see that the renormalized propagator has a complex pole with the residue free from UV divergences.<sup>3</sup> The latter property follows from the fact that the difference  $\Sigma'(s_p) - \delta Z$  is finite, because the UV divergence in  $\Sigma'(s_p)$  is equivalent to that in  $\operatorname{Re} \Sigma'(M^2)$  and the latter one is cancelled by  $\delta Z$  in any scheme. However, in the unstable-particle case, in view of non-zero  $\operatorname{Im} \Sigma'(s_p)$ , there is no way to make the residue equal to 1. Moreover, in most cases the real

part in the residue is not equal to 1, either. For instance, in the generalized by [1–3,10] OMS schemes, where the second renormalization condition is determined by the second formula in (5), one has  $1 - \delta Z + \Sigma'(s_p) = 1 + i\alpha I'_1 + \alpha^2 I_1 I''_1 + i\alpha^2 (I'_2 - I_1 R'_1) + O(\alpha^3)$ . In the  $\overline{\text{OMS}}$  scheme,  $1 - \delta Z + \Sigma'(s_p) = 1 + i\alpha I'_1 + \frac{1}{2}\alpha^2 I_1 I''_1 + i\alpha^2 (I'_2 - I_1 R'_1) + O(\alpha^3)$ .

2. The second point concerns the renormalization of the coupling constants. Formally, the renormalization prescription for coupling constants is imposed separately from that for propagators. In the electroweak theory it may be the same as in the conventionally generalized OMS scheme [3]. Namely, the  $U(1)$  constant  $e^2$  can be determined as the electric charge measured by the Compton process at the low-energy limit. The weak mixing and the weak coupling constant can be determined by relations  $s_W^2 = 1 - M_W^2/M_Z^2$  and  $g^2 = e^2/s_W^2$ , which are considered to be valid in all the orders of perturbation theory. It should be noted, however, that the actual renormalization of the coupling is determined not only by the renormalization of the coupling constant, but also by the wave-function renormalization constants of the particles participating in the interaction. So, the actual renormalized couplings, starting with the two-loop order, become different in the generalized by [1–3,10] OMS schemes and in the  $\overline{\text{OMS}}$  scheme.

3. In gauge theories considered in the framework of the renormalization scheme with the gauge-invariant renormalized masses, there is an additional constraint on the counterterms following from the gauge-invariance of bare masses. Really, the bare mass connects with the renormalized mass by means of the relation

$$M_0^2 = M^2 + (1 - \delta Z)^{-1} \delta M^2. \quad (26)$$

So, from the gauge-invariance of  $M_0^2$  and  $M^2$  the gauge-invariance of  $(1 - \delta Z)^{-1} \delta M^2$  follows. At the one-loop order this condition implies the gauge-invariance of  $R_1 \equiv \operatorname{Re} \Sigma_1(M^2)$ . Notice, due to the gauge-invariance of  $M^2$  at the one-loop order, this particular corollary is common for all the above-considered versions of the generalized OMS schemes. In case of unstable bosons in the electroweak theory this property was independently noted on the base of direct calculations [3] (it was the consequence of the consistent taking into account the tadpole contributions). At the two-loop order, in the generalized

<sup>2</sup> An equivalent normalization condition for the imaginary part of the self-energy was introduced by B. Kniehl et al. [13] for entirely different reasons, without proof that this condition fixes a renormalization scheme.

<sup>3</sup> We do not consider the IR divergences arising in the presence of massless particles. Such divergences are to be eliminated by special methods.

by [10] OMS scheme and in the  $\overline{\text{OMS}}$  scheme, the above condition implies the gauge-invariance of  $R_2 + R_1 R'_1 + I_1 I'_1$ . At the higher orders the corresponding constraints in these schemes become different.

4. In some cases the  $\overline{\text{OMS}}$  scheme is preferable with respect to the OMS scheme generalized in the sense of [10]. For instance, this is the case with unstable-particle production and decay within the two-loop precision. Really, in view of (25), the propagator in the resonance region,  $s - M^2 = O(\alpha)$ , within this precision may be approximated by the expression

$$\Delta^{-1}(s) \simeq (s - s_{p3}) \left[ 1 + \frac{1}{2} \alpha (s - s_{p3}) R''_1 \right] (\text{Res})^{-1}, \quad (27)$$

where  $\text{Res} = [1 - \delta Z + \Sigma'(s_p)]^{-1}$  is the residue in the pole (see the foregoing formulas in different schemes), and  $s_{p3}$  is the pole within the three-loop precision. In the  $\overline{\text{OMS}}$  scheme  $s_{p3}$  is determined by (23), while in the generalized by [10] OMS scheme it is determined by

$$s_{p3} = M^2 - i\alpha I_1 - i\alpha^2 I_2 - i\alpha^3 (I_3 - \frac{1}{2} I_1^2 I''_1). \quad (28)$$

Note, in both cases  $s_{p3}$  includes the  $I_3$  contribution. The common practice of taking into account the imaginary contribution to self-energy is via the unitarity relation, which relates it to the width of unstable particle at the less-loop order. However, while the width is always gauge-invariant, the imaginary part in self-energy is not always that. Really, in the generalized by [10] OMS scheme  $I_3$  is gauge-dependent, which is seen from (28) and the gauge-dependence of  $I''_1$ . At the same time, in the  $\overline{\text{OMS}}$  scheme  $I_3$  is gauge-invariant. So, in the  $\overline{\text{OMS}}$  scheme  $I_3$  can directly be related to the width of unstable particle, but not in the generalized by [10] OMS scheme.

5. The above-mentioned relation may be derived from the formula for the lifetime of an unstable particle. Below, pursuing the illustrative purposes, we present rather a heuristic derivation of this formula. So, in as much as possible idealized statement of the problem, the lifetime is directly connected with the propagator of unstable particle. Really, the amplitude of production of unstable particle (anywhere in the Universe) and its subsequent decay after the time  $x^0$ , is proportional to

$$A(x^0) \sim \int d\vec{x} \int \frac{dp}{(2\pi)^4} e^{-ipx} \Delta(p^2)$$

$$= \int \frac{dE}{2\pi} e^{-iEx^0} \Delta(E^2). \quad (29)$$

The remaining integral can be calculated with the aid of (25). By assuming parametrization  $\text{Im } s_p = M\Gamma$ , we get

$$A(x^0) \sim e^{-ix^0 M \sqrt{1 - \Gamma^2/M^2}}. \quad (30)$$

Then, the normalized-to-one probability is

$$P(x^0) = \frac{|A(x^0)|^2}{\int_0^\infty dx^0 |A(x^0)|^2} = \frac{1}{T} e^{-x^0/T}, \quad (31)$$

with  $T$  is the lifetime. The direct calculation gives

$$\frac{1}{T} = M \sqrt{2 \left( \sqrt{1 + \frac{\Gamma^2}{M^2}} - 1 \right)} = \Gamma - \frac{\Gamma^3}{8M^2} + \dots, \quad (32)$$

with dots standing for  $O(\Gamma^5/M^4)$  correction. By identifying  $T^{-1}$  with the width  $\Gamma$  of unstable particle, we derive from (32) and (24) the formula

$$I_3 = M \Gamma_{2\text{-loop}} + \Gamma_{0\text{-loop}}^3 / (8M^2), \quad (33)$$

which is valid in the  $\overline{\text{OMS}}$  scheme only. The origin of the second term in (33) may be associated with the triple cut emerging while applying the Cutkosky rules at the three-loop level.

In summary, we have constructed the  $\overline{\text{OMS}}$  scheme of UV renormalization, which we consider as most suitable for applications with unstable particles. Really, the renormalized mass in this scheme coincides with the physical mass of unstable particle, and the on-shell self-energy is directly connected with its width. Both these quantities are the observables. So, the  $\overline{\text{OMS}}$  scheme absorbs all the conveniences of the well-known complex pole scheme [9] for the parametrization of the amplitude.

The practical significance of the  $\overline{\text{OMS}}$  scheme is obvious in the case of the processes of unstable-particle production and decay considered with the two-loop (and higher) precision. Such processes are to be studied at the future colliders [14].

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